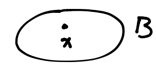
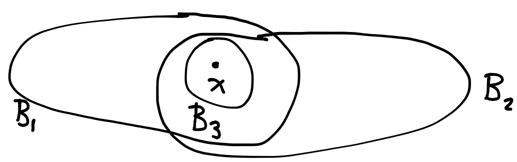
Basis for a topology

It's often difficult to describe the set of <u>all</u> open set and it's easier to describe a set whose <u>unions</u> are open.

Definition: X a set. A <u>basis</u> for a topology on X is a collection $\mathcal{B}$ of subsets of X (called <u>basis elements</u>) such that

1.) For each $x \in X$ $\exists$ $B \in \mathcal{B}$ s.t. $x \in B$. 

2.) If $x \in B_1 \cap B_2$ where $B_1, B_2 \in \mathcal{B}$, then $\exists$ a basis element B_3 s.t. $x \in B_3$ and $B_3 \subseteq B_1 \cap B_2$



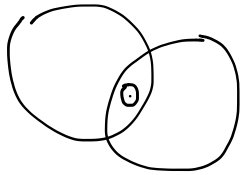
If $\mathcal{B}$ is a basis, then the <u>topology generated by $\mathcal{B}$</u> is defined as follows:

$U \subseteq X$ is open (i.e. $U \in \mathcal{T}$) if $\forall x \in U$, there is $B \in \mathcal{B}$ s.t. $x \in B$ and $B \subset U$.

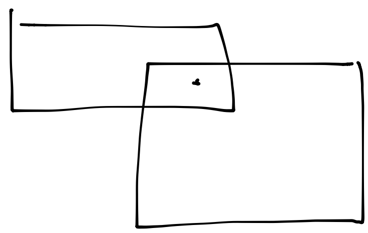
(We will check later that the topology generated by a basis is indeed a topology.)

Ex: If $\mathcal{T}$ is the standard topology on $\mathbb{R}$, then a basis for $\mathcal{T}$ is the set of open intervals $(a,b) \subseteq \mathbb{R}$ where $a<b$ (or, equivalently, the set of open balls $B_r(x)$, $x \in \mathbb{R}, r>0$).
In fact, this is how we defined the open sets in $\mathbb{R}$

Ex: Let $\mathcal{B}$ be the set of circular regions in the plane (i.e. interiors of circles). $\mathcal{B}$ is a basis for a topology on $\mathbb{R}^2$



Ex: Let $\mathcal{B}$ be the set of interiors of rectangles in $\mathbb{R}^2$. $\mathcal{B}$ is also a basis for a topology. In fact, it generates the same topology as the previous example



Ex: X a set. Let $\mathcal{B} = \{ \{x\} \mid x \in X \}$. i.e. $\mathcal{B}$ is the collection of one element subsets of X. $\mathcal{B}$ satisfies the conditions for being a basis.

What topology does $\mathcal{B}$ generate? $U \subseteq X$ is open if $\forall$ $x \in U$, there is a basis element B s.t. $x \in B \subseteq U$. But $\forall$ $U \subseteq X$ and $x \in U$, $x \in \{x\} \subseteq U$. So any subset $U \subseteq X$ is open. i.e. $\mathcal{B}$ generates the discrete topology.

Theorem: If X is a set and $\mathcal{B}$ a basis for a topology, then the topology $\mathcal{T}$ generated by $\mathcal{B}$ is, in fact, a topology.

Pf: $\emptyset \in \mathcal{T}$ vacuously. If $x \in X$, then there is a basis elt B s.t. $x \in B \subseteq X$, so $X \in \mathcal{T}$.

Let $\mathcal{T}' \subseteq \mathcal{T}$ be a collection of sets. We show $V = \bigcup_{U \in \mathcal{T}'} U \in \mathcal{T}$.
Let $x \in V$. Then $x \in U$ for some $U \in \mathcal{T}'$. so there is a basis element $B \in \mathcal{B}$ s.t. $x \in B \subseteq U$.

Thus $x \in B \subseteq V$, so $V \in \mathcal{T}$.

Now let $U_1, \dots, U_n \in \mathcal{T}$. We show that $V = U_1 \cap U_2 \cap \dots \cap U_n \in \mathcal{T}$ by induction on n.

If $n=1$, then $V = U_1 \in \mathcal{T}$.

Assume the statement holds for $n=k$.

Let $V = U_1 \cap U_2 \cap \dots \cap U_{k+1}$. Denote $W = U_1 \cap \dots \cap U_k$.

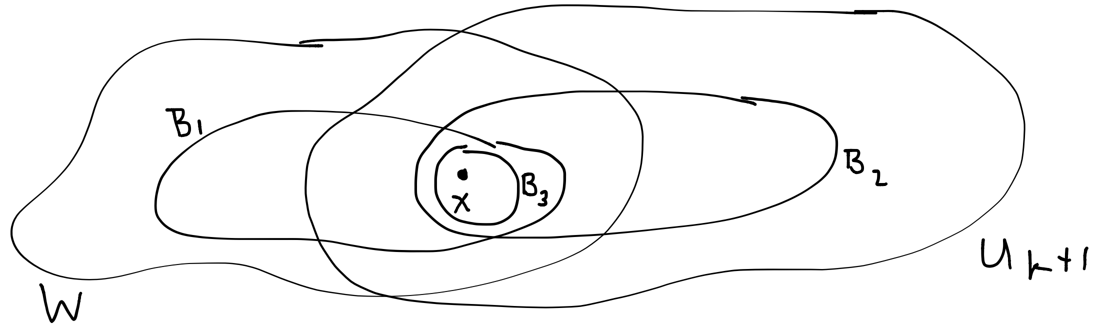
Then $V = W \cap U_{k+1}$, and by the induction hypothesis $W \in \mathcal{T}$.

Let $x \in V$. Then $x \in W$ and $x \in U_{k+1}$, so $\exists$ $B_1, B_2 \in \mathcal{B}$ s.t.

$x \in B_1 \subseteq W$ and $x \in B_2 \subseteq U_{k+1}$.

Since $\mathcal{B}$ is a basis, $\exists$ $B_3 \in \mathcal{B}$ s.t. $x \in B_3 \subseteq B_1 \cap B_2$.

Thus, $x \in B_3 \subseteq B_1 \cap B_2 \subseteq W \cap U_{k+1} = V$, so $V \in \mathcal{T}$. $\square$



Another way of describing the topology generated by a basis is as follows:

Lemma: Let X be a set. Let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ is the collection of all unions of elements of $\mathcal{B}$.

Pf: Let $\mathcal{T}'$ = the union of all collections of basis elements.

i.e. $\mathcal{T}' = \left\{ \bigcup_{B \in \mathcal{B}'} B \;\middle|\; \mathcal{B}' \subseteq \mathcal{B} \right\}$.

We want to show $\mathcal{T} = \mathcal{T}'$.

For any $B \in \mathcal{B}$, we know that if $x \in B$, then $x \in B \subseteq B$, so $B \in \mathcal{T}$ (by definition of the topology generated by a basis).

If $U \in \mathcal{T}'$, then U is a union of basis elements. Since basis elements are in $\mathcal{T}$, and $\mathcal{T}$ is a topology, $U \in \mathcal{T}$, so $\mathcal{T}' \subseteq \mathcal{T}$.

Now, let $V \in \mathcal{T}$. Then for each $x \in V$, $\exists$ a basis element B_x s.t. $x \in B_x \subseteq V$. Thus $V = \bigcup_{x \in V} B_x$, so $V \in \mathcal{T}' \Rightarrow \mathcal{T} \subseteq \mathcal{T}'$

$\Rightarrow \mathcal{T} = \mathcal{T}'$. $\square$

Ex: Consider $\mathbb{R}$. Let $\mathcal{B} = \{ [a,b) \mid a < b,\ a,b \in \mathbb{R} \}$

First, let's check that $\mathcal{B}$ is a basis.

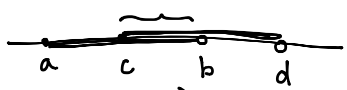
1.) If $x \in \mathbb{R}$, then $x \in [x, x+1)$

2.) If $x \in \underset{B_1}{\underset{\shortparallel}{[a,b)}} \cap \underset{B_2}{\underset{\shortparallel}{[c,d)}} = B$.

WLOG assume $a \leq c$.

If $b \leq c$, then $B = \emptyset$, so we know $b > c$.

If $b \leq d$, then $a \leq c < b \leq d$



So $B = [c, b)$, which is already a basis element.

If $d < b$, then $B = [c, d)$, which is a basis element.

Thus $\mathcal{B}$ is a basis, and the topology it generates is the union of such intervals. This is called the <u>lower limit topology</u> on $\mathbb{R}$.

<u>Ex:</u> Again we define a basis on $\mathbb{R}$. Let $K = \{\frac{1}{n} \mid n \in \mathbb{Z}_+\}$.

Let $\mathcal{B} = \{(a,b) \mid a<b\} \cup \{(a,b) - K \mid a<b\}$.

This is a basis — the intersection of two elements of $\mathcal{B}$ is again a basis element or is the empty set.

<u>Ex:</u> Consider the collection $\mathcal{B} = \{(-\infty, a) \mid a \in \mathbb{R}\}$.

This is a basis:

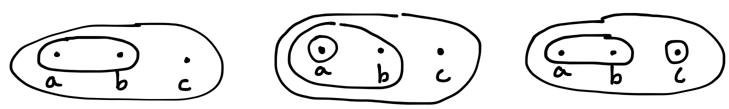
If $x \in \mathbb{R}$, $x \in (-\infty, x+1)$, and $(-\infty, a) \cap (-\infty, b) = (-\infty, \min(\{a,b\}))$, which is again a basis element.

Recall the following definition from the homework:

Def: If $\mathcal{T}_1$ and $\mathcal{T}_2$ are two topologies on X and $\mathcal{T}_1 \subseteq \mathcal{T}_2$, then $\mathcal{T}_2$ is <u>finer</u> than $\mathcal{T}_1$ and $\mathcal{T}_1$ is <u>coarser</u> than $\mathcal{T}_2$.

If neither topology is contained in the other, then $\mathcal{T}_1$ and $\mathcal{T}_2$ are <u>not comparable</u>.

Ex: Consider $X = \{a, b, c\}$



$\mathcal{T}_1$ $\mathcal{T}_2$ $\mathcal{T}_3$

$\mathcal{T}_1 \subseteq \mathcal{T}_2$ so $\mathcal{T}_2$ is finer than $\mathcal{T}_1$

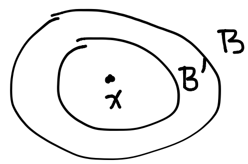
$\mathcal{T}_1 \subseteq \mathcal{T}_3$ so $\mathcal{T}_3$ is finer than $\mathcal{T}_1$.

$\{a\} \in \mathcal{T}_2 - \mathcal{T}_3$ and $\{c\} \in \mathcal{T}_3 - \mathcal{T}_2$ so $\mathcal{T}_2$ and $\mathcal{T}_3$ are not comparable.

Ex: If X is a set and $\mathcal{T}$ a topology on X, then let $\mathcal{T}' = \{\emptyset, X\}$. $\mathcal{T}' \subseteq \mathcal{T}$, so $\mathcal{T}$ is finer than $\{\emptyset, X\}$.

$\mathcal{T} \subseteq \mathcal{P}(X) =$ the discrete topology on X, so the discrete topology is finer than every other topology.

Lemma: Let $\mathcal{B}$ and $\mathcal{B}'$ be bases which generate topologies $\mathcal{T}$ and $\mathcal{T}'$ on X respectively. Then $\mathcal{T} \subseteq \mathcal{T}'$ (i.e. $\mathcal{T}'$ is finer than $\mathcal{T}$) if and only if $\forall\ x \in X$ and $B \in \mathcal{B}$ containing x, $\exists\ B' \in \mathcal{B}'$ s.t. $x \in B' \subset B$.



Pf: First assume $\mathcal{T} \subseteq \mathcal{T}'$. Let $x \in X$, and $B \in \mathcal{B}$ containing x. Then B is a basis element of $\mathcal{T}$, so $B \in \mathcal{T}$, so $B \in \mathcal{T}'$.

Thus, by definition of a topology generated by a basis, $\exists\ B' \in \mathcal{B}'$ s.t. $x \in B' \subseteq B$, as desired.

Now assume that $\forall\ x \in X$ and $B \in \mathcal{B}$ s.t. $x \in B$, $\exists\ B' \in \mathcal{B}'$ s.t. $x \in B' \subseteq B$.

We want to show $\mathcal{T} \subseteq \mathcal{T}'$. Let $U \in \mathcal{T}$. We want to show $U \in \mathcal{T}'$.

Let $x \in U$. Then $\exists\ B \in \mathcal{B}$ s.t. $x \in B \subseteq U$. Thus, $\exists\ B' \in \mathcal{B}'$ s.t. $x \in B' \subseteq B \subseteq U$. Thus, $U \in \mathcal{T}'$. $\square$

Ex: Let $\mathcal{T}$ be the standard topology on $\mathbb{R}$ and $\mathcal{T}_\ell$ the lower limit topology on $\mathbb{R}$.

$\mathcal{B} = \{(a,b) \mid a < b\}$ is a basis for $\mathcal{T}$ and

$\mathcal{B}_\ell = \{[a,b) \mid a < b\}$ is a basis for $\mathcal{T}_\ell$.

Let $x \in X$, and consider $(a,b) \in \mathcal{B}$ s.t. $x \in (a,b)$. Then $a < x < b$, so $x \in [x, b) \subseteq (a,b)$, so by the lemma, $\mathcal{T} \subseteq \mathcal{T}_\ell$.

However, consider $0 \in [0,1)$. If $0 \in (a,b)$ then $a < 0$, so $(a,b) \nsubseteq [0,1)$, so $\mathcal{T}$ is <u>not</u> finer than $\mathcal{T}_\ell$.

Note that to show that two topologies are not comparable, you should give examples in both directions (like above).